Stability Analysis of an SIR epidemic system (COVID-19)

Josephine Wairimu
PhD, Mathematical Modeling
University of Nairobi, Kenya

University of Nairobi, Kenya

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Layout

Abstract

- The aim is to study and analyze the susceptible-infectious-removed (SIR) dynamics considering the effect of health system
- The model consider a general incidence rate function and the recovery rate as functions of the number of hospital beds
- In most studies the authors assume that the recovery rate is a constant. However, in reality the recovery rate depends on time of recovering process such as the health system, including the number of hospital beds and medicines.
- The main focus of this study is analyzing the basic properties of model and demonstrating the stability properties of the model.e.

The incidence rate

- The incidence rate function f(I)S describes the mechanism of disease transmission, i.e., the rate at which susceptible become infectious
- A non linear incidence give a reasonable qualitative description of the disease dynamics
- When the 'psychological' effect is taken into the account of the infection force f(I), it is non-monotone
- This suggests that the infection force may increase when the number of infective individuals I is small while it decreases as large I increases
- For a very large number of infectives the infection force may decrease, as the number of infective individuals increases
- This may be due to the reduced number of contacts per unit time in the presence of large number of infective(Perceived fear)

The incidence rate

- The nonlinear incidence rate is generalized by the function

$$f(S, I) = \frac{\beta_1 SI}{a_1 + a_2 S + a_3 I},$$

- where the parameter a_1, a_2 and a_3 are constants
- β is the probability of transmission per contact per unit time
- a_3 measures the psychological or inhibitory effect
- The fraction $\frac{1}{a_1+a_2S+a_3I}$ measures the inhibition effect from the behavioral changes of the susceptible individuals when their number increases or from the crowding effect of the infective individuals

The Recovery rate

- The recovery rate α or the exit rate is taken as a constant in many models, in practice it depends on the time of recovering process
- It can be related to the total infectious individuals seeking treatment
- In this model, the impact of available resources of health system to the public, in particular the number of the hospital beds is incorporated.
- The recovery rate is a function of both the hospital bed-population ratio $b_1 > 0$ and the infected I.
- The hospital bed-population ratio (HBPR)- number of available hospital beds per 10,000 population used for estimating resource availability to the public(WHO).
- on the other hand recovery depends on the number of the infectious individuals I , so α is a function of b_1 and I, ie $\alpha(b_1, I)$

The Recovery rate

- The recovery rate α is given by;

$$\alpha(I) = \alpha_0 + \frac{(\alpha_1 - \alpha_0)b_1}{I + b_1},$$

- α_1 is the maximum per capita recovery rate due to the sufficient health care resource and few infectious individuals as well as the inherent property of a specific disease.
- α_0 is the minimum per capita recoveryrate due to the function of basic clinical resources.
- The medium recovery rate can be achieved when $I = b_1$, so the parameter b_1, ie the number of hospital beds plays an important role in controlling the spread of infectious diseases.
- That is the half saturation effect, $\frac{1}{2}\alpha_1$.

Hospital Bed Effect

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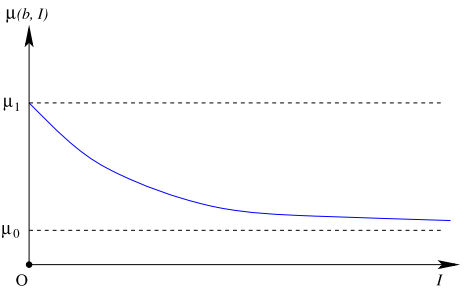

Fig. 1. The curve of recovery rate $\mu(b, I)$ with form (2.1) for a given number of hospital beds b.

Figure: Hospital Bed Population Ratio Effect

Model Differential Equations

Thus the system of differential equations is given by

$$\begin{aligned} \dot{S} &= (1-p)b - \mu_1 S - f(S,I) + \gamma R, \\ \dot{I} &= f(S,I) - (\mu_2 + \alpha(I))I, \\ \dot{R} &= pb - (\mu_3 + \gamma)R + \alpha I, \end{aligned} \tag{1}$$

- b represents the number of new borns in the population
- p represents immunized new borns thru vaccination (1-p) is the non vaccinate proportion of the new borns called vaccine efficacy) a fraction of newborn children, designated as p
- $I(t)$ is the infected population
- $R(t)$ is the recovered population, so that $N = S + I + R$.
- All the parameters are assumed to be positive

Basic Properties of the Model: Positivity

Under nonnegative conditions, the model solutions are positive.
If $S = 0$ for all $t \geq 0$, then

$$dS/dt = (1 - p)b + \gamma R \geq 0$$

If $S = N$, then, $I = 0$ and $R = 0$

$$dS/dt = (1 - p)b - \mu_1 N, \implies N* \leq \frac{(1 - p)b}{\mu_1}$$

- Therefore the susceptible population will remain in the positive othant bounded by 0 and N*
- This ensures that at any time the solution reaches the axis, its derivative increases, and the function $S(t)$ does not cross to negative part
- With can similarly prove that the $I(t)$ and $R(t)$ will remain positive for all time $t \geq 0$
- Thus, for any positive initial conditions, all equation solutions are positive.

Basic Properties of the Model: Theorem

Under nonnegative conditions, the model solutions are positive.

Theorem

Let $(S(t),I(t),R(t))$ be the solution of system of equations with initial conditions (S_0, I_0, R_0), and let $\mu = min(\mu_1, \mu_2, \mu_3)$. The compact set

$$\Omega = \{S(t), I(t), R(t) \in R_3^+, N(t) \leq \frac{b}{\mu} \quad (2)$$

is positively invariant and attracts all solutions in $\mathbb{R}_3^+$.

Basic Properties of the Model: Proof of the Theorem

Let $N(t) = S(t) + I(t) + R(t)$.Then from the system (2) we have

$$\frac{dN}{dt} \leq b - min(\mu_1, \mu_2, \mu_3)N = b - \mu)N.$$

This implies that

$$\frac{dN}{dt} + \mu N \leq b.$$

Using the method of integrating factors, we can solve and obtain the bounded region of the solution as

$$0 < N \leq \frac{b}{\mu} + (N_0 - \frac{b}{\mu})e^{-\mu t}$$

where N_0 is the initial condition. Thus $0 < N(t) < b$, as t reaches infinity, and hence Ω is a positively invariant and attractive set.

The Basic Reproduction Number

- Intuitively is the expected number of secondary infection cases caused by a single typical infective case during his entire period of infectivity in a wholly susceptible population.
- The basic reproduction number is a dimensionless quantity denoted by $\mathcal{R}_0$ and computed using the NGM as $\mathcal{R}_0 = \rho(FV^{-1})$.
- The **effective reproduction** number, or actual number of secondary infections per infectious person at any time, often denoted by R_E

$$\mathcal{R}_E = \frac{(\gamma_1 + \mu_3(1-p))b\beta_1}{a_2(\gamma_1 + \mu_3[1-p])(\alpha_1 + \mu - 2)b + a_1\mu_1(\mu_3 + \gamma_1)(\alpha_1 + \mu_2)}$$

- The infected compartments of Model (1) is I. An equilibrium solution with$I = 0$ has the form $E_0 = (S^0, 0, R^0)$, hence

$$F = [\frac{\partial \mathcal{F}_i}{\partial t}], \quad V = [\frac{\partial \mathcal{V}_i}{\partial t}]$$

- where $\mathcal{F}$, are the new infection and $\mathcal{V}$ are all the other movements in the compartment.

Equlibria

- Equating the left hand side of the model equations and solving the resulting simultaneous equations gives the solution set.
- The model has a DFE,

$$E_0(S, I, R) = \left(\frac{b(\gamma_1 + \mu_3[1-p])}{\mu_1(\mu_3 + \gamma_1)}, 0, \frac{pb}{(\mu_3 + \gamma_1)} \right)$$

- The endemic equlibrium $E_1 = (S^*, I^*, R^*)$ can be computed when $I \neq 0$, and may give more than one set of solutions due to the high order resulting polynomial in I.

Local Stability of the Disease Free equlibrium

- The Jacobian matrix of system 1 at E_0 is given by

$$J(E_0) = \begin{pmatrix} -\mu_1 & j_{12} & \gamma_1 \\ 0 & j_{22} & 0 \\ 0 & \alpha_1 & -[\mu_3 + \gamma_1] \end{pmatrix}$$

- The Eigen values of this matrix is given by

$$J(E_0) = \begin{pmatrix} -\mu_1 \\ -[\mu_3 + \gamma_1] \\ j_{22} \end{pmatrix}$$

Local Stability of the Disease Free equlibrium

- With some calculation we can show that $J_{22} = \mathcal{R}_0 - 1$.
- Therefore all the eigenvalues have negative real parts.
- So we have the following result.

Lemma

The free steady-state solution E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.

Local Stability of the Endemic equlibrium

- The Jacobian matrix of system 1 at E_1 is given by

$$J(E_1) = \begin{pmatrix} -j_{11} & j_{12} & \gamma_1 \\ j_{21} & -j_{22} & 0 \\ 0 & j_{32} & -[\mu_3 + \gamma_1] \end{pmatrix}$$

- The characteristics equation of $J(E1)$ is $\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3$ Using the Routh-Hurtwiz Criterion we can show that E_1 is locally asymptotically stable when$B_1 > 0, B_3 > 0$, and $B_1B_2B_3 > 0$. Theses conditions are satisfied under some stated conditions.
- Global stability of E_0 and E_1 can be shown by finding a suitable Lyapunov function.

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